SHARP WEIGHTED INEQUALITIES FOR MULTILINEAR FRACTIONAL MAXIMAL OPERATOR AND FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, we study the weighted inequality for multilinear fractional maximal operators and fractional integrals. We give sharp weighted estimates for both operators in some cases.

1. Introduction and Main Results

Fractional type operators and associated maximal functions are useful tools in harmonic analysis, especially in the study of differentiability or smoothness properties of functions. Recall that for $0 < \alpha < n$, the fractional maximal function M_α and the fractional integral operator I_α are defined by

$$M_\alpha(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy$$

and

$$I_\alpha(f)(x) := \int \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

respectively, where f is a locally integrable function defined on $\mathbb{R}^n$ and Q is a cube in $\mathbb{R}^n$. We refer to [6, 20] for the basic properties of these operators.

Let w be a weight, i.e., a non-negative locally integrable function. Muckenhoupt and Wheeden [18] showed that for $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, I_α is bounded from $L^p(w^p)$ to $L^q(w^q)$ if and only if w belongs to the $A_{p,q}$ class, that is,

$$[w]_{A_{p,q}} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)^q dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-p'} dx \right)^{q/p'} < \infty.$$

Moreover, the fractional maximal function M_α is also bounded from $L^p(w^p)$ to $L^q(w^q)$. See also [4, 11, 14, 19, 21, 22, 23, 24] for more results of fractional integral operators on various function spaces.

Key words and phrases. Sharp weighted inequalities; multiple weights; multilinear fractional maximal operator; multilinear fractional integrals.

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In [13], Lacey, Moen, Pérez and Torres gave the sharp weighted estimates for both M_α and I_α. Specifically, they proved that

$$\|M_\alpha\|_{L^p(w^p)\to L^q(w^q)} \le C_{n,p}[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\frac{p'}{q}} \tag{1.1}$$

and

$$\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} \le C_{n,p}[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\{1,\frac{p'}{q}\}}, \tag{1.2}$$

where the exponents of $[w]_{A_{p,q}}$ are sharp.

Multilinear fractional integral operators were studied by Grafakos [5], Kenig and Stein [12], Grafakos and Kalton [8]. For $\vec{f} = (f_1, \cdots, f_m)$ and $0 < \alpha < mn$, the multilinear fractional maximal function $\mathcal{M}_\alpha$ and the multilinear integral operator $\mathcal{I}_\alpha$ are defined by

$$\mathcal{M}_\alpha(\vec{f})(x) = \sup_{Q\ni x} \prod_{i=1}^{m} \frac{1}{|Q|^{1-\alpha/mn}} \int_Q |f_i(y_i)|dy_i$$

and

$$\mathcal{I}_\alpha(\vec{f})(x) = \int_{\mathbb{R}^{mn}} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}} d\vec{y},$$

respectively.

To study the weighted estimates for multilinear fractional integral operators, Moen [16] introduced the multiple $A_{\vec{P},q}$ weight. Let $1/p_1+\cdots+1/p_m = 1/q+\alpha/n$. A multiple weight $(w_1, \cdots, w_m)$ is said to belong to the $A_{\vec{P},q}$ class if and only if

$$[\vec{w}]_{A_{\vec{P},q}} := \sup_Q \left(\frac{1}{|Q|}\int_Q u_{\vec{w}}\right) \prod_{i=1}^{m} \left(\frac{1}{|Q|}\int_Q \sigma_i\right)^{q/p_i'} < \infty,$$

where $u_{\vec{w}} = (\Pi_{i=1}^m w_i)^q$ and $\sigma_i = w_i^{-p_i'}$. Moen showed that $\mathcal{I}_\alpha$ is bounded from $L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m})$ to $L^q(u_{\vec{w}})$ if and only if $[\vec{w}]_{A_{\vec{P},q}} < \infty$.

In this paper, we study the sharp estimates for both $\mathcal{M}_\alpha$ and $\mathcal{I}_\alpha$. For the multilinear fractional maximal function, we give a sharp estimate for $0 < \alpha < n$.

Theorem 1.1. *Suppose that $0 < \alpha < n$, $q > 0$, $1 < p_1, \cdots, p_m < \infty$, $1/p_1+\cdots+1/p_m = 1/q+\alpha/n$, $p_{i_0}' = \max\{p_i' : 1 \le i \le m\}$ for some i_0 and $p_{i_0}'(1-\alpha/n) \ge \max_{i\ne i_0}\{p_i'\}$. Let $\vec{w} = (w_1, \cdots, w_m) \in A_{\vec{P},q}$. Then we have*

$$\|\mathcal{M}_\alpha(\vec{f})\|_{L^q(u_{\vec{w}})} \le C_{m,n,\vec{P},q}[\vec{w}]_{A_{\vec{P},q}}^{(1-\frac{\alpha}{n})\frac{\max\{p_i'\}}{q}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i^{p_i})}, \tag{1.3}$$

where the exponent of $[\vec{w}]_{A_{\vec{P},q}}$ is sharp.

And for the multilinear fractional integral operators, we also get a sharp estimate in some cases.

Theorem 1.2. *Suppose that* $0 < \alpha < n$, $q > 0$, $1 < p_1, \cdots, p_m < \infty$, $1/p_1 + \cdots + 1/p_m = 1/q + \alpha/n$, $\vec{w} = (w_1, \cdots, w_m) \in A_{\vec{P},q}$ *and*

$$\min\left\{\frac{\max_i p_i'}{q}, \min_j \frac{\max_{i\neq j}\{p_i', q\}}{p_j'}\right\} \leq 1 - \frac{\alpha}{n}.$$

Then

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^q(u_{\vec{w}})} \leq C_{m,n,\vec{P},q}[\vec{w}]_{A_{\vec{P},q}}^{(1-\frac{\alpha}{n})\max_i\{1,\frac{p_i'}{q}\}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})},$$

where the exponent of $[\vec{w}]_{A_{\vec{P},q}}$ *is sharp.*

As the estimate for Calderón-Zygmund operators, although the estimate in (1.3) is sharp, it can be improved whenever mixed estimates are invoked. Moreover, we prove the sharpness for the whole scope of α.

Theorem 1.3. *Suppose that* $0 \leq \alpha < mn$, $q > 0$, $1 < p_1, \cdots, p_m < \infty$, $1/p_1 + \cdots + 1/p_m = 1/q + \alpha/n$ *and* $\vec{w} = (w_1, \cdots, w_m) \in A_{\vec{P},q}$. *Then*

$$\|\mathcal{M}_\alpha(\vec{f})\|_{L^q(u_{\vec{w}})} \leq C_{m,n,\vec{P},q}\left([\vec{w}]_{A_{\vec{P},q}}^{\frac{1}{q}} \prod_{i=1}^m [\sigma_i]_{A_\infty}^{\frac{1}{p_i}(1-\frac{\alpha p}{n})}\right) \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})},$$

where the exponents are sharp.

2. Proofs of Theorem 1.1 and Theorem 1.2

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of the cubes

$$[0, 2^{-k})^n + 2^{-k}j, \qquad k \in \mathbb{Z}, j \in \mathbb{Z}^n.$$

Denote the standard dyadic grid by $\mathcal{D}$.

By a general dyadic grid $\mathscr{D}$ we mean a collection of cubes with the following properties: (i) for any $Q \in \mathscr{D}$ its sidelength l_Q is of the form 2^k, $k \in \mathbb{Z}$; (ii) $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathscr{D}$; (iii) the cubes of a fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

For any $0 \leq \alpha < n$, we define

$$M_{\alpha,w}^{\mathscr{D}} f(x) = \sup_{Q \ni x, Q \in \mathscr{D}} \frac{1}{w(Q)^{1-\frac{\alpha}{n}}} \int_Q |f| w.$$

When $\alpha = 0$, we denote $M_{0,w}^{\mathscr{D}}$ simply by $M_w^{\mathscr{D}}$. We need the following result.

Proposition 2.1 ([17, Theorem 2.3])**.** *If* $0 \leq \alpha < n$, $1 < p \leq \frac{n}{\alpha}$ *and* $1/q = 1/p - \alpha/n$, *then*

$$\|M_{\alpha,w}^{\mathscr{D}} f\|_{L^q(w)} \leq (1 + \frac{p'}{q})^{1-\frac{\alpha}{n}} \|f\|_{L^p(w)}.$$

We say that $\mathcal{S} := \{Q_{j,k}\}$ is a sparse family of cubes if:

1. for each fixed k the cubes $Q_{j,k}$ are pairwise disjoint;
2. if $\Gamma_k = \bigcup_j Q_{j,k}$, then $\Gamma_{k+1} \subset \Gamma_k$;

(3) $|\Gamma_{k+1}\bigcap Q_{j,k}|\le \frac{1}{2}|Q_{j,k}|$.

For any $Q_{j,k}\in\mathcal{S}$, we define $E(Q_{j,k})=Q_{j,k}\setminus\Gamma_{k+1}$. Then the sets $E(Q_{j,k})$ are pairwise disjoint and $|E(Q_{j,k})|\ge\frac{1}{2}|Q_{j,k}|$.

Define

$$\mathscr{D}_t := \{2^{-k}([0,1)^n+m+(-1)^k t) : k\in\mathbb{Z}, m\in\mathbb{Z}^n\},\quad t\in\{0,1/3\}^n.$$

The importance of these grids is shown by the following proposition, which can be found in [9, proof of Theorem 1.10]. See also [15, Proposition 5.1].

Proposition 2.2. *There are 2^n dyadic grids $\mathscr{D}_t$, $t\in\{0,1/3\}^n$ such that for any cube $Q\subset\mathbb{R}^n$ there exists a cube $Q_t\in\mathscr{D}_t$ satisfying $Q\subset Q_t$ and $l(Q_t)\le 6l(Q)$.*

Given a dyadic grid $\mathscr{D}$ and a sparse family $\mathcal{S}$ in $\mathscr{D}$. Define the dyadic fractional integral operators by

$$(\mathcal{I}_\alpha^{\mathcal{S}})(\vec{f})(x)=\sum_{Q\in\mathcal{S}}|Q|^{\frac{\alpha}{n}-m}\prod_{i=1}^m\int_Q f_i(y_i)dy_i\cdot\chi_Q(x)$$

and

$$(\mathcal{I}_{\alpha,q}^{\mathcal{S}})(\vec{f})(x)=\left(\sum_{Q\in\mathcal{S}}\left(|Q|^{\frac{\alpha}{n}-m}\prod_{i=1}^m\int_Q f_i(y_i)dy_i\right)^q\cdot\chi_Q(x)\right)^{1/q}.$$

In [16], Moen showed that for $q>1$

$$(\mathcal{I}_\alpha(\vec{f}))(x)\le C\sum_{Q\in\mathscr{D}}\frac{l(Q)^\alpha}{|3Q|^m}\int_{(3Q)^m}f_1(y_1)\cdots f_m(y_m)d\vec{y}\chi_Q(x),$$

and for $q\le 1$,

$$(\mathcal{I}_\alpha(\vec{f}))(x)\le C\left(\sum_{Q\in\mathscr{D}}\left(\frac{l(Q)^\alpha}{|3Q|^m}\int_{(3Q)^m}f_1(y_1)\cdots f_m(y_m)d\vec{y}\right)^q\chi_Q(x)\right)^{1/q}.$$

By Proposition 2.2 and similar arguments as that in [2], we get

$$\mathcal{I}_\alpha(\vec{f})(x)\lesssim\sum_{t\in\{0,1/3\}^n}(\mathcal{I}_\alpha^{\mathcal{S}_t})(\vec{f})(x),\quad q>1,\tag{2.1}$$

and

$$\mathcal{I}_\alpha(\vec{f})(x)\lesssim\sum_{t\in\{0,1/3\}^n}(\mathcal{I}_{\alpha,q}^{\mathcal{S}_t})(\vec{f})(x),\quad q\le 1.\tag{2.2}$$

We have the following lemmas.

Lemma 2.3. *Suppose that $\vec{w}=(w_1,\cdots,w_m)\in A_{\vec{P},q}$, $0\le\alpha<mn$, $1<q,p_1,\cdots,p_m<\infty$ and $1/p_1+\cdots+1/p_m=1/q+\alpha/n$. Then*

$$\vec{w}^i := (w_1,\cdots,w_{i-1},(\Pi_{i=1}^m w_i)^{-1},w_{i+1},\cdots,w_m)\in A_{\vec{P}^i,p_i'},$$

where $\vec{P}^i=(p_1,\cdots,p_{i-1},q',p_{i+1},\cdots,p_m)$ and $[\vec{w}^i]_{A_{\vec{P}^i,p_i'}}=[\vec{w}]_{A_{\vec{P},q}}^{p_i'/q}$.

Proof. By the definition we have

$$
\begin{aligned}
[\vec{w}^i]_{A_{\vec{P}^i,p_i'}} &= \sup_Q \left(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\right)\left(\frac{1}{|Q|}\int_Q u_{\vec{w}}\right)^{p_i'/q} \prod_{j\neq i}\left(\frac{1}{|Q|}\int_Q w_j^{-p_j'}\right)^{p_i'/p_j'} \\
&= [\vec{w}]_{A_{\vec{P},q}}^{p_i'/q}.
\end{aligned}
$$

□

Lemma 2.4. *Suppose that* $0<\alpha<n$, $1<p_1,\cdots,p_m<\infty$, $1/p_1+\cdots+1/p_m=1/q+\alpha/n$, $q(1-\alpha/n)\geq max\{p_i'\}$ *and that* $\vec{w}=(w_1,\cdots,w_m)\in A_{\vec{P},q}$. *Then*

$$
\|\mathcal{I}_\alpha^{\mathcal{S}}(|f_1|,\cdots,|f_m|)\|_{L^q(u_{\vec{w}})} \leq C_{m,n,\vec{P},q}[\vec{w}]_{A_{\vec{P},q}}^{1-\alpha/n}\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}.
$$

Proof. It is equivalent to prove the following

$$
\|\mathcal{I}_\alpha^{\mathcal{S}}(|f_1|\sigma_1,\cdots,|f_m|\sigma_m)\|_{L^q(u_{\vec{w}})} \leq C_{m,n,\vec{P},q}[\vec{w}]_{A_{\vec{P},q}}^{1-\alpha/n}\prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}.
$$

Let $1/p=1/q+\alpha/n$. We have

$$
\begin{aligned}
(2.3)\quad & \|\mathcal{I}_\alpha^{\mathcal{S}}(|f_1|\sigma_1,\cdots,|f_m|\sigma_m)\|_{L^q(u_{\vec{w}})} \\
= \quad & \sup_{\|h\|_{L^{q'}(u_{\vec{w}})}=1} \int_{\mathbb{R}^n} \mathcal{I}_\alpha^{\mathcal{S}}(|f_1|\sigma_1,\cdots,|f_m|\sigma_m) h u_{\vec{w}} \\
= \quad & \sup_{\|h\|_{L^{q'}(u_{\vec{w}})}=1} \sum_{Q\in\mathcal{S}} |Q|^{\frac{\alpha}{n}-m}\prod_{i=1}^m \int_Q |f_i|\sigma_i dy_i \cdot \int_Q h u_{\vec{w}} \\
= \quad & \sup_{\|h\|_{L^{q'}(u_{\vec{w}})}=1} \sum_{Q\in\mathcal{S}} \left(\frac{u_{\vec{w}}(Q)}{|Q|}\right)^{1-\frac{\alpha}{n}} \prod_{i=1}^m \left(\frac{\sigma_i(Q)}{|Q|}\right)^{\frac{q(1-\alpha/n)}{p_i'}} \prod_{i=1}^m \sigma_i(Q)^{1-\frac{q}{p_i'}(1-\frac{\alpha}{n})} \\
& \times |Q|^{(m-\frac{\alpha}{n})((1-\frac{\alpha}{n})q-1)} \cdot \frac{1}{u_{\vec{w}}(Q)^{1-\alpha/n}}\int_Q h u_{\vec{w}} \cdot \prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q |f_i|\sigma_i dy_i \\
\leq \quad & [\vec{w}]_{A_{\vec{P},q}}^{1-\alpha/n} \sup_{\|h\|_{L^{q'}(u_{\vec{w}})}=1} \sum_{Q\in\mathcal{S}} |Q|^{(m-\frac{\alpha}{n})\frac{q}{p'}} \prod_{i=1}^m \sigma_i(Q)^{1-\frac{q}{p_i'}(1-\frac{\alpha}{n})} \\
& \times \frac{1}{u_{\vec{w}}(Q)^{1-\alpha/n}}\int_Q h u_{\vec{w}} \cdot \prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q |f_i|\sigma_i dy_i.
\end{aligned}
$$

By Hölder's inequality, we have

$$
(2.4)\qquad |E(Q)| = \int_{E(Q)} u_{\vec{w}}^{\frac{1}{(m-\frac{\alpha}{n})q}} \prod_{i=1}^m \sigma_i^{\frac{1}{(m-\frac{\alpha}{n})p_i'}} dx
$$

$$\leq \quad u_{\vec{w}}(E(Q))^{\frac{1}{(m-\frac{\alpha}{n})q}} \prod_{i=1}^{m} \sigma_i(E(Q))^{\frac{1}{(m-\frac{\alpha}{n})p_i'}}.$$

Since $|Q| \leq 2|E(Q)|$ and $q(1-\alpha/n) \geq \max\{p_i'\}$, we have

$$\begin{aligned}
&|Q|^{(m-\frac{\alpha}{n})\frac{q}{p'}} \prod_{i=1}^{m} \sigma_i(Q)^{1-\frac{q}{p_i'}(1-\frac{\alpha}{n})} \\
\lesssim\ & u_{\vec{w}}(E(Q))^{\frac{1}{p'}} \prod_{i=1}^{m} \sigma_i(E(Q))^{\frac{q}{p'p_i'}} \prod_{i=1}^{m} \sigma_i(Q)^{1-\frac{q}{p_i'}(1-\frac{\alpha}{n})} \\
\leq\ & u_{\vec{w}}(E(Q))^{\frac{1}{p'}} \prod_{i=1}^{m} \sigma_i(E(Q))^{1/p_i}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{Q\in\mathcal{S}} |Q|^{(m-\frac{\alpha}{n})\frac{q}{p'}} \prod_{i=1}^{m} \sigma_i(Q)^{1-\frac{q}{p_i'}(1-\frac{\alpha}{n})} \frac{1}{u_{\vec{w}}(Q)^{1-\alpha/n}} \int_Q h u_{\vec{w}} \\
&\times \prod_{i=1}^{m} \frac{1}{\sigma_i(Q)} \int_Q |f_i|\sigma_i dy_i \\
\lesssim\ & \sum_{Q\in\mathcal{S}} \frac{1}{u_{\vec{w}}(Q)^{1-\alpha/n}} \int_Q h u_{\vec{w}} \cdot u_{\vec{w}}(E(Q))^{\frac{1}{p'}} \cdot \prod_{i=1}^{m} \frac{1}{\sigma_i(Q)} \int_Q |f_i|\sigma_i dy_i \\
&\times \prod_{i=1}^{m} \sigma_i(E(Q))^{1/p_i} \\
\leq\ & \left(\sum_{Q\in\mathcal{S}} \left(\frac{1}{u_{\vec{w}}(Q)^{1-\alpha/n}} \int_Q h u_{\vec{w}} \right)^{p'} u_{\vec{w}}(E(Q)) \right)^{1/p'} \\
&\times \prod_{i=1}^{m} \left(\sum_{Q\in\mathcal{S}} \left(\frac{1}{\sigma_i(Q)} \int_Q |f_i|\sigma_i dy_i \right)^{p_i} \sigma_i(E(Q)) \right)^{1/p_i} \\
\leq\ & \|M^{\mathscr{D}}_{\alpha,u_{\vec{w}}} h\|_{L^{p'}(u_{\vec{w}})} \prod_{i=1}^{m} \|M^{\mathscr{D}}_{\sigma_i} f_i\|_{L^{p_i}(\sigma_i)} \\
\lesssim\ & \|h\|_{L^{q'}(u_{\vec{w}})} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\sigma_i)}.
\end{aligned}$$

It follows from (2.3) that

$$\|\mathcal{I}^{\mathcal{S}}_\alpha(|f_1|\sigma_1, \cdots, |f_m|\sigma_m)\|_{L^q(u_{\vec{w}})} \leq C_{m,n,\vec{P},q} [\vec{w}]^{1-\alpha/n}_{A_{\vec{P},q}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\sigma_i)}.$$

□

Lemma 2.5. *Let the hypotheses be as in Theorem 1.1. Moreover, let $\mathcal{S}$ be a sparse family of cubes. Then we have*

$$\|\mathcal{I}^{\mathcal{S}}_{\alpha,q}(|f_1|,\cdots,|f_m|)\|_{L^q(u_{\vec{w}})} \le C_{m,n,\vec{P},q}[\vec{w}]_{A_{\vec{P},q}}^{(1-\frac{\alpha}{n})\max_i\{\frac{p_i'}{q}\}}\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}.$$

Proof. Without loss of generality, assume that $p_1=\min\{p_1,\cdots,p_m\}$. As in Lemma 2.4, it is equivalent to prove the following

$$\|\mathcal{I}^{\mathcal{S}}_{\alpha,q}(|f_1|\sigma_1,\cdots,|f_m|\sigma_m)\|_{L^q(u_{\vec{w}})} \le C_{m,n,\vec{P},q}[\vec{w}]_{A_{\vec{P},q}}^{(1-\alpha/n)\max\{p_i'/q\}}\prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}.$$

We have

$$\begin{aligned}
&\|\mathcal{I}^{\mathcal{S}}_{\alpha,q}(|f_1|\sigma_1,\cdots,|f_m|\sigma_m)\|^q_{L^q(u_{\vec{w}})}\\
= & \sum_{Q\in\mathcal{S}}\left(|Q|^{\frac{\alpha}{n}-m}\prod_{i=1}^m\int_Q |f_i|\sigma_i dy_i\right)^q\cdot u_{\vec{w}}(Q)\\
= & \sum_{Q\in\mathcal{S}}\left(\frac{u_{\vec{w}}(Q)}{|Q|}\right)^{(1-\frac{\alpha}{n})p_1'}\prod_{i=1}^m\left(\frac{\sigma_i(Q)}{|Q|}\right)^{\frac{(1-\alpha/n)qp_1'}{p_i'}}|Q|^{q(m-\frac{\alpha}{n})((1-\frac{\alpha}{n})p_1'-1)}\\
&\times\prod_{i=2}^m\sigma_i(Q)^{q-\frac{qp_1'}{p_i'}(1-\frac{\alpha}{n})}u_{\vec{w}}(Q)^{1-(1-\alpha/n)p_1'}\left(\frac{\sigma_1(Q)^{\frac{\alpha}{n}}}{\sigma_1(Q)}\int_Q|f_1|\sigma_1dy_1\right)^q\\
&\times\prod_{i=2}^m\left(\frac{1}{\sigma_i(Q)}\int_Q|f_i|\sigma_i dy_i\right)^q\\
\lesssim & [\vec{w}]_{A_{\vec{P},q}}^{(1-\alpha/n)p_1'}\sum_{Q\in\mathcal{S}}\left(\frac{\sigma_1(Q)^{\frac{\alpha}{n}}}{\sigma_1(Q)}\int_Q|f_1|\sigma_1dy_1\right)^q\prod_{i=2}^m\left(\frac{1}{\sigma_i(Q)}\int_Q|f_i|\sigma_i dy_i\right)^q\\
&\times|E(Q)|^{q(m-\frac{\alpha}{n})((1-\frac{\alpha}{n})p_1'-1)}u_{\vec{w}}(Q)^{1-(1-\alpha/n)p_1'}\prod_{i=2}^m\sigma_i(Q)^{q-\frac{qp_1'}{p_i'}(1-\frac{\alpha}{n})}\\
\le & [\vec{w}]_{A_{\vec{P},q}}^{(1-\alpha/n)p_1'}\sum_{Q\in\mathcal{S}}\left(\frac{\sigma_1(Q)^{\frac{\alpha}{n}}}{\sigma_1(Q)}\int_Q|f_1|\sigma_1dy_1\right)^q\prod_{i=2}^m\left(\frac{1}{\sigma_i(Q)}\int_Q|f_i|\sigma_i dy_i\right)^q\\
&\times\sigma_1(E(Q))^{q((1-\frac{\alpha}{n})-\frac{1}{p_1'})}\prod_{i=2}^m\sigma_i(E(Q))^{q/p_i},
\end{aligned}$$

where we use (2.4) and the fact that $p_1'(1-\alpha/n)\ge\max_{i\neq1}\{p_i'\}$ in the last step.

Let $1/q_1=1/p_1-\alpha/n$. By Hölder's inequality, we have

$$\|\mathcal{I}^{\mathcal{S}}_{\alpha,q}(|f_1|\sigma_1,\cdots,|f_m|\sigma_m)\|^q_{L^q(u_{\vec{w}})}$$

$$\begin{aligned}
&\le [\vec{w}]_{A_{\vec{P},q}}^{(1-\alpha/n)p_1'} \left(\sum_{Q\in\mathcal{S}} \left(\frac{\sigma_1(Q)^{\alpha/n}}{\sigma_1(Q)} \int_Q |f_1|\sigma_1 dy_1 \right)^{q_1} \sigma_1(E(Q)) \right)^{q/q_1} \\
&\quad \times \prod_{i=2}^m \left(\sum_{Q\in\mathcal{S}} \left(\frac{1}{\sigma_i(Q)} \int_Q |f_i|\sigma_i dy_i \right)^{p_i} \sigma_i(E(Q)) \right)^{q/p_i} \\
&\le [\vec{w}]_{A_{\vec{P},q}}^{(1-\alpha/n)p_1'} \|M_{\alpha,\sigma_1}^{\mathscr{D}} f_1\|_{L^{q_1}(\sigma)}^q \prod_{i=2}^m \|M_{\sigma_i}^{\mathscr{D}} f_i\|_{L^{p_i}(\sigma_i)}^q \\
&\lesssim [\vec{w}]_{A_{\vec{P},q}}^{(1-\alpha/n)p_1'} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}^q,
\end{aligned}$$

where $p_1'(1-\frac{\alpha}{n}) > 1$ ensure that $p_1 < \frac{n}{\alpha}$. □

Proof of Theorem 1.1. By Proposition 2.2, we have

$$\mathcal{M}_\alpha(f_1,\cdots,f_m)(x) \le C_{m,n} \sum_{t\in\{0,1/3\}^n} \mathcal{M}_\alpha^{\mathscr{D}_t}(f_1,\cdots,f_m)(x).$$

So it suffices to prove the desired conclusion for $\mathcal{M}_\alpha^{\mathscr{D}}$. Let $a = 2^{(m-\alpha/n)(n+1)}$ and $\Omega_k = \{x\in\mathbb{R}^n : \mathcal{M}_\alpha^{\mathscr{D}}(f_1,\cdots,f_m)(x) > a^k\}$. Suppose that $\Omega_k = \bigcup_j Q_j^k$, where Q_j^k are pairwise disjoint maximal dyadic cubes in Ω_k. Then $\mathcal{S} := \{Q_j^k\}$ is a sparse family. To see this, it suffices to prove that

$$|Q_j^k \bigcap \Omega_{k+1}| \le \frac{1}{2}|Q_j^k|.$$

In fact, by the definition of Q_j^k, we have

$$a^k < \prod_{i=1}^m \frac{1}{|Q_j^k|^{1-\alpha/mn}} \int_{Q_j^k} |f_i| \le 2^{mn-\alpha} a^k.$$

It follows that

$$\begin{aligned}
|Q_j^k \bigcap \Omega_{k+1}| &= \sum_{Q_l^{k+1}\subset Q_j^k} |Q_l^{k+1}| \\
&\le \sum_{Q_l^{k+1}\subset Q_j^k} a^{-(k+1)/(m-\alpha/n)} \prod_{i=1}^m \Big(\int_{Q_l^{k+1}} f_i \Big)^{1/(m-\alpha/n)} \\
&\le a^{-(k+1)/(m-\alpha/n)} \prod_{i=1}^m \left(\sum_{Q_l^{k+1}\subset Q_j^k} \Big(\int_{Q_l^{k+1}} f_i \Big)^{1/(1-\alpha/nm)} \right)^{1/m} \\
&\le a^{-(k+1)/(m-\alpha/n)} \prod_{i=1}^m \left(\int_{Q_j^k} f_i \right)^{1/(m-\alpha/n)} \\
&\le a^{-1/(m-\alpha/n)} 2^n |Q_j^k|
\end{aligned}$$

$$= \frac{1}{2}|Q_j^k|.$$

Hence $\mathcal{S} := \{Q_j^k\}$ is a sparse family. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{M}_\alpha^{\mathscr{D}}(f_1,\cdots,f_m)^q u_{\vec{w}} dx \\
= &\sum_k \int_{\Omega_k \setminus \Omega_{k+1}} \mathcal{M}_\alpha^{\mathscr{D}}(f_1,\cdots,f_m)^q u_{\vec{w}} dx \\
\le &\ a^q \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|^{1-\alpha/mn}} \int_{Q_j^k} |f_i(y_i)| dy_i \right)^q u_{\vec{w}}(E(Q_j^k)) \\
\lesssim &\int_{\mathbb{R}^n} (\mathcal{I}_{\alpha,q}^{\mathcal{S}})(f_1,\cdots,f_m)^q u_{\vec{w}} dx
\end{aligned} \tag{2.5}$$

Now the desired conclusion follows from Lemma 2.5. □

Proof of Theorem 1.2. There are two cases.

(i). $q > 1$. By (2.1), it suffices to prove that

$$\|\mathcal{I}_\alpha^{\mathcal{S}}(\vec{f})\|_{L^q(u_{\vec{w}})} \le C_{m,n,\vec{P},q}[\vec{w}]_{A_{\vec{P},q}}^{(1-\frac{\alpha}{n})\max_i\{1,\frac{p_i'}{q}\}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}.$$

If $q(1-\alpha/n) \ge max\{p_i'\}$, the desired conclusion follows from Lemma 2.4. If $p_j'(1-\frac{\alpha}{n}) \ge \max_{i\neq j}\{p_i', q\}$, without loss of generality, assume that $p_1'(1-\frac{\alpha}{n}) \ge \max_{i\neq 1}\{p_i', q\}$. By duality, we have

$$\begin{aligned}
&\|\mathcal{I}_\alpha^{\mathcal{S}}\|_{L^{p_1}(w_1^{p_1})\times\cdots\times L^{p_m}(w_m^{p_m})\to L^q(u_{\vec{w}})} \\
= &\|\mathcal{I}_\alpha^{\mathcal{S}}\|_{L^{q'}(\Pi_{i=1}^m w_i^{-q'})\times L^{p_2}(w_2^{p_2})\times\cdots\times L^{p_m}(w_m^{p_m})\to L^{p_1'}(w_1^{-p_1'})} \\
\lesssim &[\vec{w}]_{A_{\vec{P},q}}^{(1-\frac{\alpha}{n})\frac{p_1'}{q}},
\end{aligned}$$

where we use Lemma 2.3 in the last step.

(ii). $q \le 1$. In this case,

$$\min_j\{\frac{\max_{i\neq j}\{p_i'\}}{p_j'}\} = \min\left\{\frac{\max_i p_i'}{q}, \min_j \frac{\max_{i\neq j}\{p_i', q\}}{p_j'}\right\} \le 1 - \frac{\alpha}{n}.$$

By (2.2), it suffices to prove that

$$\|\mathcal{I}_{\alpha,q}^{\mathcal{S}}(\vec{f})\|_{L^q(u_{\vec{w}})} \le C_{m,n,\vec{P},q}[\vec{w}]_{A_{\vec{P},q}}^{(1-\frac{\alpha}{n})\max_i\{1,\frac{p_i'}{q}\}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}.$$

By Lemma 2.5, we get

$$\|\mathcal{I}_{\alpha,q}^{\mathcal{S}}(\vec{f})\|_{L^q(u_{\vec{w}})} \le C_{m,n,\vec{P},q}[\vec{w}]_{A_{\vec{P},q}}^{(1-\frac{\alpha}{n})\max_i\{\frac{p_i'}{q}\}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}$$

$$= C_{m,n,\vec{P},q}[\vec{w}]_{A_{\vec{P},q}}^{(1-\frac{\alpha}{n})\max_i\{1,\frac{p_i'}{q}\}}\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}.$$

This completes the proof. □

3. Proof of Theorem 1.3

First, we introduce the sharp reverse Hölder's property of A_∞ weights which was proved in [9] and [10, Theorem 2.3]. Recall that

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)}\int_Q M(w\chi_Q).$$

Proposition 3.1 ([9, Theorem 2.3]). *Let $w \in A_\infty$. Then*

$$\left(\frac{1}{|Q|}\int_Q w^{r(w)}\right)^{1/r(w)} \le 2\frac{1}{|Q|}\int_Q w, \tag{3.1}$$

where $r(w) = 1+\frac{1}{\tau_n[w]_{A_\infty}}$ and $\tau_n = 2^{11+n}$. Notice that the conjugate $r(w)' \simeq [w]_{A_\infty}$.

We also need the following characterization of $A_{\vec{P},q}$ weights.

Proposition 3.2 ([16, Theorem 3.4]). *Suppose that $1 < p_1, \cdots, p_m < \infty$, and $\vec{w} \in A_{\vec{P},q}$. Then*

$$u_{\vec{w}} \in A_{mq} \quad \text{and} \quad \sigma_i \in A_{mp_i'}.$$

Proof of Theorem 1.3. Set $\alpha_i = (p_i'r_i)'$, where r_i is the exponent in the sharp reverse Hölder's inequality (3.1) for the weights σ_i which are in A_∞ for $i = 1$, $\cdots$, m. By (2.5), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{M}_\alpha^{\mathscr{D}}(f_1,\cdots,f_m)^q u_{\vec{w}}dx \\
\le\;& \sum_{k,j}\left(\prod_{i=1}^m \frac{1}{|Q_j^k|^{1-\alpha/mn}}\int_{Q_j^k}|f_i(y_i)|dy_i\right)^q u_{\vec{w}}(E(Q_j^k)) \\
\le\;& \sum_{k,j}\frac{u_{\vec{w}}(E(Q_j^k))}{|Q_j^k|^{-\alpha q/n}}\prod_{i=1}^m\left(\frac{1}{|Q_j^k|}\int_{Q_j^k}|f_i|^{\alpha_i}w_i^{\alpha_i}\right)^{\frac{q}{\alpha_i}}\cdot\left(\frac{1}{|Q_j^k|}\int_{Q_j^k}w_i^{-\alpha_i'}\right)^{\frac{q}{\alpha_i'}} \\
\le\;& \sum_{k,j}\frac{u_{\vec{w}}(E(Q_j^k))}{|Q_j^k|^{-\alpha q/n}}\prod_{i=1}^m\left(\frac{1}{|Q_j^k|}\int_{Q_j^k}|f_i|^{\alpha_i}w_i^{\alpha_i}\right)^{\frac{q}{\alpha_i}}\cdot\left(\frac{2}{|Q_j^k|}\int_{Q_j^k}\sigma_i\right)^{\frac{q}{p_i'}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Proposition 3.1)} \\
\le\;& C[\vec{w}]_{A_{\vec{P},q}}\sum_{k,j}|E(Q_j^k)|\prod_{i=1}^m\left(\frac{|Q_j^k|^{\frac{\alpha p\alpha_i}{np_i}}}{|Q_j^k|}\int_{Q_j^k}|f_i|^{\alpha_i}w_i^{\alpha_i}\right)^{\frac{q}{\alpha_i}}
\end{aligned}$$

$$
\begin{aligned}
&\le C[\vec{w}]_{A_{\vec{P},q}} \int_{\mathbb{R}^n} \prod_{i=1}^m M^{\mathscr{D}}_{\frac{\alpha p \alpha_i}{p_i}}(|f_i|^{\alpha_i} w_i^{\alpha_i})^{q/\alpha_i} dx \\
&\le C[\vec{w}]_{A_{\vec{P},q}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} M^{\mathscr{D}}_{\frac{\alpha p \alpha_i}{p_i}}(|f_i|^{\alpha_i} w_i^{\alpha_i})^{q_i/\alpha_i} dx \right)^{q/q_i},
\end{aligned}
$$

where

$$\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha p}{n p_i}, \quad i = 1, \cdots, m.$$

Substituting $\alpha p \alpha_i / p_i$ for α and letting $w = 1$ in Proposition 2.1, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{M}^{\mathscr{D}}_\alpha(f_1, \cdots, f_m)^q u_{\vec{w}} dx \\
\le\ & C[\vec{w}]_{A_{\vec{P},q}} \prod_{i=1}^m \left(1 + \frac{(p_i/\alpha_i)'}{(q_i/\alpha_i)}\right)^{(1-\frac{\alpha p \alpha_i}{n p_i})\frac{q}{\alpha_i}} \left\| |f_i|^{\alpha_i} w_i^{\alpha_i} \right\|^{\frac{q}{\alpha_i}}_{L^{\frac{p_i}{\alpha_i}}(\mathbb{R}^n)} \\
=\ & C[\vec{w}]_{A_{\vec{P},q}} \prod_{i=1}^m \left(1 + \frac{(p_i/\alpha_i)'}{(q_i/\alpha_i)}\right)^{(1-\frac{\alpha p \alpha_i}{n p_i})\frac{q}{\alpha_i}} \|f_i\|^q_{L^{p_i}(w_i^{p_i})}.
\end{aligned}
$$

It is easy to check that $\frac{p_i}{\alpha_i} - 1 \simeq [\sigma_i]^{-1}_{A_\infty}$. Therefore

$$1 + \frac{(p_i/\alpha_i)'}{(q_i/\alpha_i)} \lesssim 1 + \frac{p_i}{q_i}[\sigma_i]_{A_\infty} \lesssim [\sigma_i]_{A_\infty}$$

and

$$
\begin{aligned}
(1 - \frac{\alpha p \alpha_i}{n p_i})\frac{q}{\alpha_i} &= \frac{q}{p_i} - \frac{\alpha p q}{n p_i} + \frac{q}{p_i}(\frac{p_i}{\alpha_i} - 1) \\
&\le \frac{q}{p_i}\left(1 - \frac{\alpha p}{n}\right) + \frac{q}{p_i} \cdot C[\sigma_i]^{-1}_{A_\infty}.
\end{aligned}
$$

Consequently,

$$\left(1 + \frac{(p_i/\alpha_i)'}{(q_i/\alpha_i)}\right)^{(1-\frac{\alpha p \alpha_i}{n p_i})\frac{q}{\alpha_i}} \lesssim [\sigma_i]^{\frac{q}{p_i}(1-\frac{\alpha p}{n})}_{A_\infty}.$$

Now we get

$$\|\mathcal{M}_\alpha(\vec{f})\|_{L^q(u_{\vec{w}})} \le C_{m,n,\vec{P},q} \left([\vec{w}]^{\frac{1}{q}}_{A_{\vec{P},q}} \prod_{i=1}^m [\sigma_i]^{\frac{1}{p_i}(1-\frac{\alpha p}{n})}_{A_\infty} \right) \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}.$$

□

4. Examples

Finally, we end with some examples to show that our bounds are sharp. First we show that Theorem 1.1 is sharp. Consider the case $m = 2$ (we leave it to the reader to modify the example for $m > 2$) and suppose that

$$\|\mathcal{M}_\alpha\|_{L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2}) \to L^q(w_1^q w_2^q)} \lesssim [\vec{w}]^{r(1-\frac{\alpha}{n})\max(\frac{p_1'}{q}, \frac{p_2'}{q})}_{A_{\vec{P},q}} \tag{4.1}$$

for some $r < 1$. Further suppose that $p_1' \geq p_2'$. For $0 < \varepsilon < 1$, let $f_1(x) = |x|^{\varepsilon - n}\chi_{B(0,1)}(x)$, $f_2(x) = |x|^{\frac{\varepsilon-n}{p_2}}\chi_{B(0,1)}(x)$, $w_1(x) = |x|^{(n-\varepsilon)/p_1'}$ and $w_2(x) = 1$. Calculations show that $\|f_1\|_{L^{p_1}(w_1^{p_1})} \simeq \varepsilon^{-1/p_1}$, $\|f_2\|_{L^{p_2}(w_2^{p_2})} \simeq \varepsilon^{-1/p_2}$, and $[\vec{w}]_{A_{\vec{P},q}} \simeq \varepsilon^{-q/p_1'}$. For $x \in B(0,1)$ we have

$$\begin{aligned}\mathcal{M}_\alpha(f_1,f_2)(x) &\gtrsim \frac{1}{|x|^{n-\frac{\alpha}{2}}}\int_{B(0,|x|)}|y_1|^{\varepsilon-n}\,dy_1 \cdot \frac{1}{|x|^{n-\frac{\alpha}{2}}}\int_{B(0,|x|)}|y_2|^{\frac{\varepsilon-n}{p_2}}\,dy_2 \\ &\gtrsim \frac{1}{\varepsilon}|x|^{\varepsilon-n+\frac{\varepsilon-n}{p_2}+\alpha}.\end{aligned}$$

Hence,

$$\begin{aligned}(4.2)\ \|\mathcal{M}_\alpha(f_1,f_2)\|_{L^q(w_1^q w_2^q)} &\gtrsim \frac{1}{\varepsilon}\Big(\int_{B(0,1)}|x|^{(\varepsilon-n)(q+\frac{q}{p_2}-\frac{q}{p_1'})+\alpha q}\,dx\Big)^{1/q} \\ &\simeq \frac{1}{\varepsilon}\Big(\int_0^1 x^{(1+\frac{\alpha q}{n})\varepsilon-1}\,dx\Big)^{1/q} \\ &\simeq \frac{1}{\varepsilon}\Big(\frac{1}{\varepsilon}\Big)^{1/q}.\end{aligned}$$

Combining this with inequality (4.1) we see for some $r < 1$, $\left(\frac{1}{\varepsilon}\right)^{1+\frac{1}{q}} \lesssim \left(\frac{1}{\varepsilon}\right)^{r(1-\frac{\alpha}{n})+\frac{1}{p}}$, which is impossible as $\varepsilon \to 0$.

Next we show that Theorem 1.2 is sharp. It is easy to notice that $\mathcal{M}_\alpha \leq C_{m,n,\alpha}\mathcal{I}_\alpha$. If $\max_i p_i' \geq q$, then using the same f_i and w_i as above we get (4.2) with $\mathcal{M}_\alpha$ replaced by $\mathcal{I}_\alpha$, showing the sharpness. On the other hand, if $\max_i p_i' < q$, the sharpness follows from the standard duality argument used in the proof of Theorem 1.2.

Finally we show that Theorem 1.3 is sharp. For $0 < \varepsilon < 1$, let

$$w_i(x) = |x|^{(n-\varepsilon)/p_i'} \quad \text{and} \quad f_i(x) = |x|^{\varepsilon-n}\chi_{B(0,1)}(x), \quad i = 1, \cdots, m.$$

Then $u_{\vec{w}} = |x|^{(n-\varepsilon)(m-\frac{1}{p})q}$ and it is easy to check that

$$[\vec{w}]_{A_{\vec{p},q}} \simeq \left(\frac{1}{\varepsilon}\right)^{q(m-\frac{1}{p})}, \quad [\sigma_i]_{A_\infty} \lesssim \frac{1}{\varepsilon} \quad \text{and} \quad \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})} \simeq \left(\frac{1}{\varepsilon}\right)^{1/p}.$$

For $x \in B(0,1)$, we have

$$\begin{aligned}\mathcal{M}_\alpha(\vec{f})(x) &\gtrsim \prod_{i=1}^m \frac{1}{|x|^{n-\frac{\alpha}{m}}}\int_{B(0,|x|)}|y_i|^{\varepsilon-n}\,dy_i \\ &\gtrsim \left(\frac{1}{\varepsilon}\right)^m |x|^{m(\varepsilon-n)+\alpha}.\end{aligned}$$

Therefore,

$$\|\mathcal{M}_\alpha(\vec{f})\|_{L^q(u_{\vec{w}})} \gtrsim \left(\frac{1}{\varepsilon}\right)^m \left(\int_{B(0,1)}|x|^{mq(\varepsilon-n)+\alpha q+(n-\varepsilon)(m-\frac{1}{p})q}dx\right)^{\frac{1}{q}}$$

$$\begin{aligned}
&\simeq \left(\frac{1}{\varepsilon}\right)^m \left(\int_0^1 t^{(\alpha+1)\varepsilon-1}dt\right)^{\frac{1}{q}} \\
&\simeq \left(\frac{1}{\varepsilon}\right)^{m+\frac{1}{q}} \\
&\gtrsim \left([\vec{w}]_{A_{\vec{P},q}}^{\frac{1}{q}} \prod_{i=1}^m [\sigma_i]_{A_\infty}^{\frac{1}{p_i}(1-\frac{\alpha p}{n})}\right) \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}.
\end{aligned}$$

References

[1] X. Chen and Q. Xue, Weighted estimates for a class of multilinear fractional type operators, J. Math. Anal. Appl., 362 (2010), 355–373.

[2] D. Cruz-Uribe and K. Moen, One and two weight norm inequalities for Riesz potentials, Illinois J. Math., In Press.

[3] D. Cruz-Uribe and K. Moen, A fractional Muckenhoupt-Wheeden theorem and its consequences, http://arxiv.org/abs/1303.3424

[4] Y. Ding, S. Lan, Fractional integral operators on anisotropic Hardy spaces, Integral Equations Operator Theory, 60 (2008), 329–356.

[5] L. Grafakos, On multilinear fractional integrals, Studia Math., 102 (1992), 49–56.

[6] L. Grafakos, Classical Fourier Analysis, Second Edition, Springer-Verlag, 2008.

[7] L. Grafakos, Modern Fourier Analysis, Second Edition, Springer-Verlag, 2008.

[8] L. Grafakos and N. Kalton, Some remarks on multilinear maps and interpolation, Math. Ann., 319 (2001), 151–180.

[9] T. Hytönen and C. Pérez, Sharp weighted bounds involving A_∞, Analysis & PDE, (to appear).

[10] T. Hytönen, C. Pérez and E. Rela, Sharp reverse Hölder property for A_∞ weightes on spaces of homogeneous type, J. Funct. Anal., 263 (2012), 3883–3899.

[11] T. Iida, E. Sato,Y. Sawano, and H. Tanaka, Sharp bounds for multilinear fractional integral operators on Morrey type spaces, Positivity, 16 (2012), 339–358.

[12] C.E. Kenig and E.M. Stein, Multilinear estimates and fractional integration, Math. Res Lett., 6 (1999), 1–15.

[13] M.T. Lacey, K. Moen, C. Pérez and R.H. Torres, Sharp weighted bounds for fractional integral operators, J. Funct. Anal., 259 (2010), 1073–1097.

[14] S. Lan, M. Lee and C. Lin, Fractional integral operators on weighted anisotropic Hardy spaces, J. Operator Theory, 68 (2012), 3–17.

[15] A. Lerner, On an estimate of Calderón-Zygmund operators by dyadic positive operators, J. Anal. Math., to appear.

[16] K. Moen, Weighted inequalities for multilinear fractional integral operators, Collect. Math., 60 (2009), 213–238.

[17] K. Moen, Sharp weighted bounds without testing or extrapolation, Archiv der Mathematik, 99 (2012) 457–466.

[18] B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for fractional integrals, Trans. Amer. Math. Soc. 192 (1974), 261–274.

[19] E.T. Sawyer and R.L. Wheeden, Weighted inequlities for fractional integrals on Euclidean and homogeneous spaces, Amer. J. Math. 114 (1992), 813–874.

[20] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, NJ, 1970.

[21] W. Wang, C. Tan and Z. Lou, A note on weighted norm inequalities for fractional maximal operators with non-doubling measures, Taiwanese J. Math., 16 (2012), 1409–1422.

[22] Characterizations of localized BMO(R^n) via commutators of localized Riesz transforms and fractional integrals associated to Schrödinger operators, Collect. Math., 61 (2010), 65–79.

[23] Da. Yang, Do. Yang and Y. Zhou, Endpoint properties of localized Riesz transforms and fractional integrals associated to Schrödinger operators, Potential Anal, 30 (2009), 271–300.

[24] Y. Zhong and J. Chen, Modulation space estimates for the fractional integral operators, Sci. China Math., 54 (2011), 1479–1489.

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